CHAPTER 2

Developing Understanding of Measurement

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Measurement is an enterprise that spans both mathematics and science yet has its roots in everyday experience. Most of us can probably recollect instances from childhood when we wondered why objects look smaller as we walk away from them or why their appearance changes with transitions in perspective. Later, we recast **and** comprehend these everyday experiences by modeling **and** measuring aspects of space so that measures of triangles, coupled with assumptions about light, explain **the** changes wrought by transitions in perspective (Gravemeijer, 1998). I recall puzzling about how the announcers of the Mercury rocket launches at Cape Canaveral knew the height of a rocket in flight and wondering how I might know the height my model rocket attained at its apogee, even without benefit of a sk tape measure. My speculations had their practical side as well. My grandfather was a carpenter who thought every young apprentice should know not only the measure on one side of his six-foot extending ruler 1 1/2 feet) but also its counterpart on the opposite side (e.g., 4 1/2 feet). My recollections encompass measure's dual qualities of practical grasp and imaginative reach. On the one hand, to measure is to do. On the other hand, to measure is to imagine qualities of the world such as length and time.

This synthetic character of measure is readily apparent in its history. For example, Eratosthenes estimated the circumference of the earth nearly 2,200 years ago by both imagining and doing. He imagined or assumed that the earth was spherical, that the rays of the sun could be considered to be parallel, and that the form of a circle approximated the surface of the earth. As a matter of practicality, he knew that in a well located at Syene, sunlight penetrated all the way to the bottom. He also knew **that** this absence of shadow meant that light followed a **line** like that depicted in Figure 12.1. By practical measure, he knew that Alexandria, located approximately due north of Syene, was 5,000 stades away. A stick at Alexandria cast a shadow with an angle measure of 1/50 of the arc of a circle (about 7 1/2 degrees). So this meant that the circumference of the earth must be 250,000 stades, or about 25,000 miles, not too far from the modern estimate. Eratosthenes also appreciated the potential error in his measure, adjusting it because Alexandria is not quite due north from Syene.

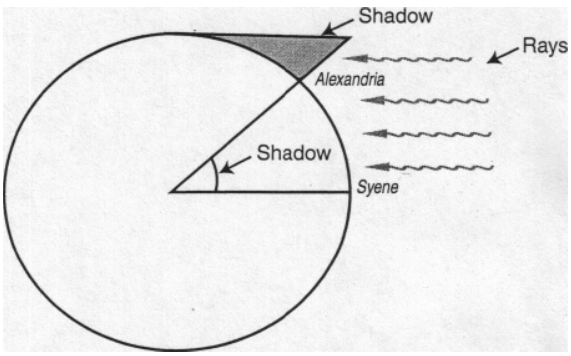

Figure 12.1. Eratosrhenes approximated the circumference of the earth by using shadows.

The investigation conducted by Eratosthenes illustrates a deep connection between modeling space and exploring its extent. Measuring space requires the construction of a model to represent it and also tools that embody and extend the model. Measurement is inherently imprecise. Sources of imprecision include model-world mismatches, measurement-device qualities, and observer qualities (Kerr & Lester, 1976).

Ironically, because of Euclid's emphasis on straightedge and compass constructions, geometry is often treated separately from measurement. This approach has some virtues, especially when it leads to generalizations that go beyond the natural world. Yet measurement has traditionally served as one route for developing mathematical knowledge of space (Kline, 1959). A pedagogical implication of this history is that learning to measure qualities of space, such as length, area, volume, and angle, provides a practical means for developing fundamental understandings of its structure. For these reasons, substantial pedagogical value is gained in realigning measure and space. Moreover, conceptions involved in measuring space readily extend to other phenomena, such as mass or time, as well as to relational measures, such as rate. These extensions often prove crucial to the development of scientific reasoning (Lehrer & Schauble, 2000).

Because most of the research to date focuses on children's conceptions of spatial measure, the review reflects this research concentration. The first section of this chapter describes components of a mathematics of measure from the perspective of child development. This developmental perspective is influenced by Piageifs studies of important transitions in children's conceptions of spatial measure (Piaget, Inhelder, & Szeminska, 1960) and also by related research guided by other traditions (e.g., Davydov, 1975; Miller, 1984). In the sections that follow, I employ frameworks from this developmental literature to organize a summary of children's evolving understandings of length, area, volume, and angle measure. These progressions are based primarily on two kinds of evidence: clinical interviews conducted with children at various ages or grades and longitudinal study of transitions in children's reasoning during the course of traditional schooling.

More recent work examines the acquisition of measure concepts in classrooms that emphasize guided reinvention of the underpinnings of measure rather than simple procedural competence. This contemporary work suggests the need to revise previous accounts of development to include consideration of the mediational means (e.g., Wertsch, 1998), including forms of mathematical notation and argument that are employed by teachers and students during the course of instruction about measurement. For example, as I subsequently describe more fully, children's sense of length measure as a paced distance evolves when they shift from the plane of activity to representing their paces as a ruler that uses their footsteps as the units of measure. In the final section of this chapter, I summarize research that extends the investigation of conceptions of measure in other directions, including student thinking about the nature and sources of error of measurement.

Understanding Measure: A Developmental Perspective

Much of the research in the field draws on the seminal contributions of Piaget (Piaget & Inhelder, 1948/1956; Piaget et al., 1960), which continue to be a wellspring for contemporary research. Piaget's analysis suggested that conceptions of spatial measure were not unitary but instead consisted of a web of related constructs leading to the eventual construction and coordination of standard units. Piaget was careful to distinguish between activity, such as using a ruler, and reflective abstraction on activity, such as understanding the role played by the identical units in the ruler. In Piaget's view, understanding of measure entailed a successive mental restructuring of space, so that conceptions of measure increasingly encompassed subdivisions of space and translations of these subdivisions to comprise a measurement.

Piaget and his colleagues further suggested that conceptual change was tightly coupled with the overall development of reasoning. Accordingly, conservation (recognition of invariance under transformation, undergirded by such mental operations as reversibility, a form of mental undoing) of length, area, volume, and angle was a hallmark of, and constraint on, development in each domain of spatial measure studied by Piaget. Measurement was also considered to be tightly coupled with quantity, so that measurement was an outgrowth of counting. However, studies conducted in the past two decades generally fail to support this tight coupling of the development of understanding of spatial measure with quantity or even with general capacities for mental logic. For example, Hiebert (1981a, 1981b) conducted an instructional study in which first-grade children, some of whom conserved length and some of whom did not, were taught important underpinnings of length measure, such as iterating units (accumulating units by counting) to measure lengths. Hiebert found that acquiring ideas like iteration was generally unrelated to a child's status as a conserver. The sole component related to conservation was recognition of the inverse relationship between the length of a unit of measure and the resulting count (e.g.. smaller units produce larger counts). In a like vein. Miller (1984) noted that even preschool children generally employed systematic procedures to ensure equal distributions of snacks (measured via lengths and areas) when solving problems requiring spontaneous measurement procedures. The general lack of relationship between conservation and understanding of measure 1$ also characteristic of other domains of spatial measure (Carpenter, 1975).

Studies generally fail to support Piaget's claims that general forms of logic strongly constrain children's ideas about measure. These findings imply that little value is **wined** by delaying or withholding instruction until a chiild is mentally "ready" to learn about measurement. In contrast, it has proved useful to consider conceptual changed about measure as change in a network or web of ideas related to unit. Some of the most prominent conceptual foundations include the following:

(1) Unit-attribute relations. Correspondence between units and the attribute being measured must be established. Although this relationship may seem transparent, especially in light of the ubiquity of tools like rulers and protractors, children in fact often misappropriate units of length measure for the measurement of other spatial extent, such as area, volume, and, perhaps most notoriously, angle. The suitability of a particular unit of measure usually involves a trade-off between models of the space being measured and the tools that are practical for the purposes at hand. For example, estimating the area of a playground might involve decisions about a model of it, such as a parallelogram, and the purposes for which the measure is intended. Should the measure be made to the nearest square inch? Square foot? In a similar vein, consider the distance between two cities. Can units of time (e.g., it's 2 1/2 hours between Madison and Chicago) be used instead of units of length to measure distance? If so, what assumptions need to be made?

(2) Iteration. Units can be reused; this understanding is based on subdivision (to establish congruent parts) and translation. For example, to iterate a unit of length, a child must come to understand length as a distance that can be subdivided. Moreover, these subdivisions can be accumulated and, if necessary, rearranged to measure a length. Hence, given a fixed length eight units long and a single unit, one measures the length by subdividing it into eight congruent partitions. This task is accomplished by translating the unit successively from the start point to the end point.

(3) Tiling. Units fill lines, planes, volumes, and angles. For example, to measure a length, one needs to arrange units in succession. Young children often find it useful to lay units in succession but sometimes are unaware of the consequences of leaving "cracks." If so, using counts of units as representations of length is problematic. Tiling (space-filling) is implied by subdivision of lengths, areas, volumes, and angles, but this implication is not transparent to all children.

(4) Identical units. If the units are identical, a count will represent the measure. Mixtures of units should be explicitly marked, for example, as "5 yards and 3 inches," not "8."

(5) Standardization. Conventions about units facilitate communication. Though arbitrary, standard units often have interesting histories. For example, Nickerson (1999) suggests that the relation of 12 inches to a foot likely arose from the confluence of several related developments. First, the ratio of the spans of thumb and foot is usually about 12, an important ratio for measurement centered on readily available parts of the body. Such a ratio would have had particular advantages before the advent of mass-produced tools. Second, many fractions of 12, like one half or one third, are integers, making subdivision relatively convenient. Third, 12 is the sum of 3, 4, and 5 and thus can be used to make perpendicular joints with beams: Mark 3 units along one beam from the end at which the joint will be made, mark 4 along the other, and then adjust the angle so that the 5-unit beam touches the ends of both lengths simultaneously.

(6) Proportionality. Measurements with different-sized units imply that different quantities can represent the same measure. These quantities will be inversely proportional to the size of the units. Consequently, a foot-long strip has a measure of 12 inches, or 24 half-inches.

(7) Additivity. Units of Euclidean space can be decomposed and recomposed, so that, for example, the total distance between two points is equivalent to the sum of the distances of any arbitrary set of segments that subdivide the line segment. For example, if B is any point on the segment AC, then $AB + BC = AC$. This recognition is implicit in studies of conservation, in which, for example, the length of an object is not affected by its translation to a new point. Similarly, the lengths of two paths may be different even if they begin and end at the same points on the plane, because the sum of the parts of one path exceeds the sum of the parts of another. The recognition of this component of spatial measure is often a significant intellectual milestone.

(8) Origin (zero-point). Measurement often involves the development of a scale. Although scale properties vary with different systems of measure, measures of Euclidean space conform to ratios, so that, for example, the distance between 0 and 10 is the same as that between 30 and 40. This conformity implies that any location on the scale can serve as the origin. Other common forms of measurement may not have these properties. For example, when judging how much one likes chocolate ice cream on a five-point scale, it is difficult to

know whether a judgment of 4 indicates twice as much affection as a judgment of 2.

Collective coordination among these eight components constitutes an informal theory of measure. Studies conducted in the past two decades suggest that children's developing sense of measurement is marked by gradual coordination and consolidation of these components. Research has been oriented toward tracking transitions across a profile and range of understandings rather than a unitary construct of measurement. For example, a child might be able to subdivide a line, yet fail to appreciate the function of identical subdivisions. Children's theories, of course, have a limited scope and precision when contrasted with those developed by mathematicians and scientists. Nevertheless, understanding these constituents of measure and their relations establishes a firm ground for future exploration of the mathematics of measure, and their acquisition also implies coming to understand the (Euclidean) structure of space.

In the following sections, I review developmental progressions of children's understandings of length, area, volume, and angle measure with an eye toward documenting important milestones in children's notions of the components of measure noted previously. The review is intended to be representative, not exhaustive. Two kinds of studies are included: cognitive development studies and classroom studies. Studies of cognitive development typically engage groups of children in activities designed to reveal how they think or understand an issue. Children at different ages (cross-sectional) are compared, or the same children are followed for a period of time, (longitudinal) to observe transitions in thinking. These studies provide glimpses of children's thinking under conditions of activity and learning that are typically found in the culture. In contrast, classroom studies modify instruction and then investigate the effects of these modifications on children's thinking or understanding. The resulting portraits of student reasoning are not always in close agreement with those obtained from studies of cognitive development. One reason may be that the cultural experiences of measure are less frequent and perhaps less thought-provoking than those deliberately created in the design of instruction.

Length Measure

Studies of Cognitive Development

Length measure builds on preschoolers' understanding that lengths span distances (e.g., Miller & Baillargeon, 1990). Measure of distances requires restructuring space so that one "sees" counts of units as representing an iteration of successive distances. Iteration refers to accumulating units of measure to obtain a quantity, such as 12 inches. It rests on a foundation of subdividing length and ordering the subdivisions (Piaget et al., 1960). Thus, a count of *n* units represents a distance of *n* units. Studies of children's development suggest that acquisition of this understanding involves the coordination of multiple constructs, especially those of unit and zero-point. As noted previously, the construction of unit involves a web of foundational ideas including procedures of iteration, recognition of the need for identical units, understanding of the inverse relationship between magnitude of each unit and the resulting length measure, and understanding of partitions of unit. Understanding zero-point involves the mental coordination of the origin and endpoint of the scale used to measure length, so that the length from the 10 cm mark on the scale to the 20 cm mark is considered equivalent to that between 2 cm and 12 cm. Developmental studies indicate that these constructs are not acquired in an all-or-none manner, nor are they necessarily tightly linked. Most studies suggest that these understandings of units of length are acquired over the course of the elementary grades, although significant variations in developmental trajectories occur when different forms of instruction are employed.

Children's first understandings of length measure often involve direct comparison of objects (Lindquist, 1989; Piaget et al., 1960). Congruent objects have equal lengths, and congruency is readily tested when objects can be superimposed or juxtaposed. Yet young children (first grade) also typically understand that they can compare the length of two objects by representing the objects with a string or paper strip (Hiebert, 1981a, 1981b). This use of representational means likely draws on experiences of objects "standing for" others in early childhood, such as in pretend play in which a banana can represent a telephone yet retain its identity as a fruit (Leslie, 1987) or in the ability to distinguish, yet coordinate, models or pictures and their referents (deLoache, 1989). Moreover, first graders can use given units to find the length of different objects, and they associate higher counts with longer objects (Hiebert, 1981a, 1981b, 1984). Most young children (first and second graders) even understand that counts of smaller units will be larger than counts of larger units, given the same length (Carpenter & Lewis, 1976; Lehrer, Jenkins, & Osana, 1998b).

These understandings of the practical use of units are probably grounded in childhood experiences in which children observe others use rulers and related measurement devices and incorporate the resulting lessons learned into their play. However, facility with counting not imply understanding of length measure as a metric distance. Because early measure understanding es as a collection of developing concepts, children understand qualities of measure, such as the inverse mediation between counts and size of units, yet not fully appreciate other constituents of length measure, such as function of identical units or the operation of iteration of unit (Lehrer et al., 1998b). These concepts are much more problematic for primary-grade children (i.e., axes 6 to 8). Children often have difficulty creating units of equal size (Miller, 1984), and even when provided equal units, first and second graders often do not understand their purposes, so they freely mix, for example, Inches and centimeters, counting all to "measure" a length (Lehrer et al., 1998b). For these students, measure is not significantly differentiated from counting (Hatano & Ito, 1965). For example, younger students in the study by Lehrer , Jenkins, and Osana (1998b) often imposed their thumbs, pencil erasers, or other invented units on a length, counting each but failing to attend to inconsistencies among these invented units (and often mixing their inventions with other units). Even given identical units, significant minorities of young children fail to spontaneously iterate units of measure when they "run out" of units, despite demonstrating procedural competence with rulers (Hatano & Ito, 1965; Lehrer et al., 1998b). For example, given 8 units and a 12-unit length, some of these children lay the 8 units end to end and then decide that they cannot proceed further. They cannot conceive of how one could reuse any of the 8 units, perhaps because they have not mentally subdivided the remaining space into unit partitions. Moreover, young children (e.g., first grade or kindergarten) may coordinate some of the components of iteration, such as use of units of constant size and repeated application, yet not others, such as tiling. For example, first and second graders may leave "spaces" between identical units even as they repeatedly use a single unit to "measure" a length (Horvath & Lehrer, 2000; Koehler & Lehrer, 1999).

Children's understanding of zero-point is particularly tenuous. Only a minority of young children understand that any point on a scale can serve as the starting point, and even a significant minority of older children (e.g., fifth grade) respond to nonzero origins by simply reading off whatever number on a ruler aligns with the end of the object (Lehrer et al., 1998a). Many children throughout schooling begin measuring with one rather than zero (Ellis, Siegler, & Van Voorhis, 2001). Parts of units create additional complexities. For example, Lehrer, Jacobson, Kemeny, & Strom (1999) noted that some second-grade children (7 to 8 years old) measured a 2 and 1/2-units strip of paper by counting "1, 2, [pause], 3 [pause], 3 and a half." They explained that the 3 referred to the third unit counted, but "there's only a half," so in effect the last unit was represented twice, first as a count of unit and then as a partition of a unit. Yet these same children could readily coordinate different starting and ending points for integers (e.g., starting at 3 and ending at 7 yielded the same measure as starting at 1 and ending at 5).

Classroom Studies

Recent work has focused on establishing developmental trajectories for understanding of linear measure in classrooms that promote representation and communication. These studies suggest that important gains are realized in understanding when children's learning is mediated by systems of inscription (e.g., what children write) and notation (Greeno & Hall, 1997). For example, Clements, Battista, and Sarama (1998) reported that using computer tools that mediated children's experience of unit and iteration helped children mentally restructure lengths into units. Other recent studies place a premium on making transitions from embodied forms of length measure, such as pacing, to inscribing and symbolizing these forms as *foot strips* and other kinds of measurement tools (Lehrer et al., 1999; McClain, Cobb, Gravemeijer, & Estes, 1999). Inscriptions like foot strips help children reason about the mathematically important components of activity (e.g., the lengths spanned while pacing) so that paces are transformed into units of measure. By constructing tools, children have the opportunity to discover the measurement principles that guide the design of these tools. Although constructing and using tools have a long tradition in teaching practice, recent studies of mediated activity provide important details about how these practices contribute to conceptual change (e.g., Wertsch, 1998). Generally, asking children to represent their experiences tends to help them select and make visible mathematically important components of activity. For example, when pacing, mathematically fruitful components include "lifting out" paces as units that can be iterated to obtain a length measure and deciding what is meant by "walking straight." Other elements of the activity, such as maintaining one's balance while pacing, are placed in the (mathematical) background.

Further work in classrooms suggests the importance of providing opportunities for children to repeatedly "split" (Confrey, 1995; Confrey & Smith, 1995), or partition, lengths to come to understand unit partitions. For example, in second-grade classrooms where students had the opportunity to design rulers, students were motivated by their previous experience with rulers to add "marks" that would help them measure lengths that were parts of units, for instance, 3 1/2. To develop these unit partitions, children folded their unit (represented as a length of paper strip) in half, then repeated this process to create fourths, eighths, and even sixty-fourths (Lehrer et al., 1999). In addition to developing procedures for partitioning, children were able to examine the resulting partitions (inscribed as fold lines) by unfolding the resulting strips so that they could design their own ruler marks. Eventually, these actions helped children develop their first understandings of operator conceptions of fractions (e.g., half of half of half, etc.).

Classrooms are excellent forums for understanding the importance of conventional units. For example, if children measure the length of objects with different units, such questions as "Which object is longest?" may direct attention to the communicative functions of standard units. Classroom studies also point to creative ways of melding measure and the study of form in the elementary grades in ways that recall their historical co-development. For example, children in Elizabeth Penner's first- and second-grade class searched for forms (e.g., lines, triangles, squares) that would model the configuration of players in a fair game of tag (Penner & Lehrer, 2000). Attempts to model the shape of fairness initiated cycles of exploration involving length measure and properties related to length in each form (e.g., distance from the corners of a square to the center). Eventually, children decided that circles were the fairest of all forms because the locus of points defining a circle was equidistant from its center. This insight was achieved by developing understanding of linear measure and by employing this understanding to explore the properties of shape and form. Such tight coupling between space and measure is reminiscent of Piaget's investigations but helps one .to see these linkages as objects of instructional design rather than as pre-existing qualities of mind.

Area Measure

Cognitive Developmental Studies

In many ways, studies of children's conceptions of area measure parallel those discussed for length. I focus here on a longitudinal investigation of 37 children in grades 1 through 3 who were followed for 3 years, because the results of this study are representative of much of the literature (Lehrer et al., 1998b). Young children (e.g., in first and second grades) often treat length measure as a surrogate for area measure. For example, Lehrer et al. (1998b) found that some young children measured the area of a square by measuring the length of a side, then moving the ruler over a bit and measuring the length between the sides again, and so on, treating length as a space-filling attribute. When provided manipulatives (i.e., squares, right triangles, circles, and rectangles) for use in finding the area measure of a variety of forms, most children in grades 1 through 3 freely mixed units and then reported the total count of the units. The two most commonly observed strategies with use of manipulatives were *boundedness* and *resemblance*. That is, children deployed units in ways that would not violate the boundaries of closed figures, and they often used units that resembled the figure being measured (e.g., triangles for triangles). Young children were also liable to ignore the space-filling properties of units, preferring instead to honor the boundaries of the forms, so when presented with a choice between "leaving cracks" and overlapping a boundary, they invariably chose the former. Figure 12.2 displays two second-grade students' approaches to measuring the "space covered" by their hands (Lehrer et al., 1998a). The solution labeled "a" uses beans as units of measure, and the one labeled "b" uses spaghetti. Both units were chosen because they "looked like" (resembled) the contour of the hand. Both solutions also ignore the "cracks" (space-filling). A third solution, proposed by their teacher and labeled "c," consisted of an overlay of square units of measure. The class initially rejected this solution, both because the squares "went over" the outline of the hand (boundedness) and because the squares "looked wrong" (did not resemble the contours of the hand).

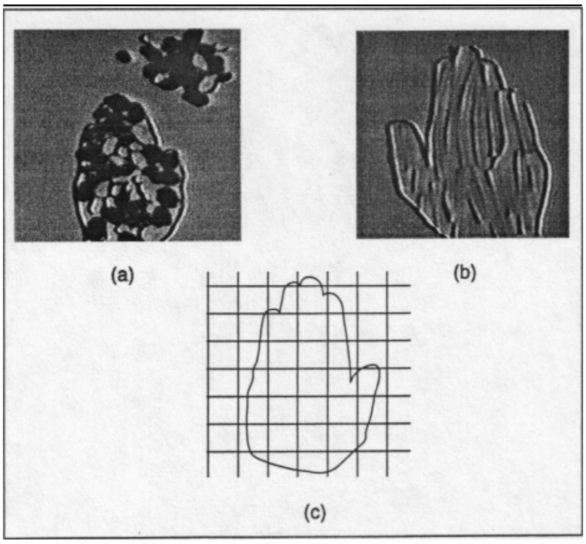

Figure 12.2. Students proposed measures using (a) beans, (b) spaghetti, and (c) an overlay of square units to measure the space covered by their hands,

Over the elementary grades, area measure becomes differentiated from length measure, and the space-filling (tiling) property of the unit becomes more apparent to most children (Lehrer et al., 1998a). However, other aspects of area measure remain problematic, even though students can recall standard formulas for finding the areas of squares and rectangles. Fewer than 20% of the students in the study by Lehrer et al. (1998b) believed that area measure required identical units, and fewer than half could reconfigure a series of planar figures so that known area measures could be used to find the measures of the areas of unknown figures. Just as linear measure requires restructuring a length into a succession of distances, area measure requires restructuring of the plane. Consequently, students found it very difficult to decompose and then recompose the areas of forms to see one form as a composition of others.

Similarly, Battista, Clements, Arnoff, Battista, and Can Auken Borrow (1998) reported that students in the primary grades often cannot structure a rectangle as an array of units. In a more extensive exploration of children's strategies for structuring rectangular arrays across grades 1 through 4, Outhred and Mitchelmore (2000) found a wide range of conceptions of array. Students were asked, for example, to find the number of 1-cm squares needed to cover a 6-cm by 5-cm rectangle. Many first and second graders either incompletely covered the rectangle or drew coverings that varied the size of the square unit. Third graders were more successful in using concrete units to cover the rectangle, but most did not use the structure of the rows of the array to accomplish this task, relying instead on perception. Only in the fourth grade did the majority of students use the dimensional structure of the array to measure it. These students used one dimension to find the number of units in each row and used the other to find the number of rows. Nevertheless, a significant minority (about 30%) of these older students simply tried to count squares using the more primitive strategies. These findings are especially troublesome in light of the widespread use of area models of fractions and the use of array models for multiplication, which apparently assume knowledge that may not be in place.

In sum, conceptual development in area measure lagged behind that of length measure. Understanding core conceptual notions, such as identical units and tiling, was typical of students by the end of the elementary grades for length measure but not for area measure. Younger children often employed resemblance as the prime criterion for selecting a unit of area measure, suggesting the need for attention to the qualities of unit that make it suitable for area measure. Other studies focus on students' conceptions of the area measure of rectangles. Most often, rectangular area is treated in schooling as a simple matter of multiplying lengths, but the research suggests that many students in the elementary grades do not "see" this product as a measurement. Many fail to structure even a simple form like a rectangle as an array that could conceivably be measured with unit squares. Current practices of giving students squares as units may lead to apparent procedural competence but fail to challenge students' preconceptions about what makes a unit suitable. Moreover, many students understand square units as things to be counted rather than as subdivisions of the plane.

Classroom Studies

As with length measure, studies of developmental trajectories of area measure in classrooms that emphasize representation and communication reveal significant departures from patterns typically described in the literature. For example, Lehrer et al. (1998a) found that an effective way for second-grade students to begin working on ideas about area was to solve problems involving partitioning and reallotment of areas without measuring. In the course of this partitioning and rearranging, students came to regard one of the partitions as a unit so that counts of this unit afforded ready comparison among areas. Later, children explored the suitability of different units (e.g., beans) for finding the areas of irregular forms, such as handprints, and found that units like squares had the desirable properties of space-filling and identity. By the end of the school year, these children had little difficulty creating two-dimensional arrays of units for rectangles. For example, one student's solution to the question of whether the areas of 5 x 8 and 4 x 10 rectangles (with unlabeled dimensions) were the same or different is displayed in Figure 12.3. This student first used length measure to partition each rectangle into unit squares and then demonstrated that skip-counting by columns or by rows of either resulted in the same count (40). He also concluded that if he rotated the rectangle, he could readily "see" commutativity because "5 X 8 = 8 X 5."

These second graders often spontaneously imposed these kinds of arrays on nonpolygonal forms to find approximate solutions to area. For example, one student's solution to finding the space covered by her hand is displayed in Figure 12.4. She used color regions to represent approximate partitioning of units, such as one fourth or one third. She collected these parts of units and added them to the units completely enclosed by the figure

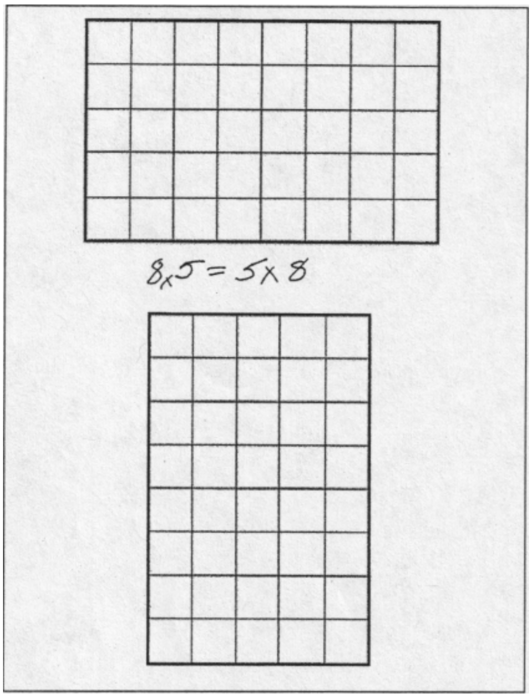

Figure 12.3. A student's visual demonstration of commutativity of multiplication.

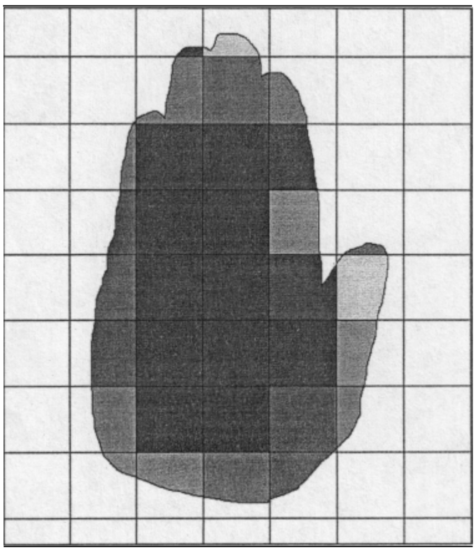

Figure 12.4. A **second grader represents parts** of units **with different colors and combines like-colored regions to approximate whole units.**

to arrive at an estimate of the area of her hand. These findings suggest that mental restructuring of units of arrays is assisted by classroom emphasis on representation and argument. The developmental patterns noted so often in the literature probably reflect the shortcomings of typical experiences, including instruction.

Volume Measure

The measure of volume presents some additional complexities for reasoning about the structure of space, primarily because units of measure must be defined and coordinated in three dimensions. The research conducted often blends classroom study with description of individual change, so this section and the one that follows on angle measure reflect this synthesis.

An emerging body of work addresses the strategies that students employ to structure a volume, given a unit. For example, Battista and Clements (1998) noted a range of strategies employed by students in the third and fifth grades to mentally structure a three-dimensional array of cubes. Many students, especially the younger ones, could count only the faces of the cubes, resulting in frequent instances of multiple counts of a single cubic unit and a failure to count any cubic units in the interior of the cube. The majority of fifth-grade students, but only about 20% of third-grade students, structured the array as a series of layers. Layering enabled students to count the number of units in one layer and then multiply or skip-count to obtain the total number of cubic units in the cube. These findings suggest that, as with area and length, students' models of spatial structure influence their conceptions of its measure.

Classroom studies again suggest that forms of representation heavily influence how students conceive of structuring volume. For example, third-grade students with a wide range of experiences and representations of volume measure structured space as three-dimensional arrays. Unlike the younger students described in the Battista and Clements (1998) study, all could structure cubes as three-dimensional arrays. Most even came to conceive of volume as a product of area and height (Lehrer, Strom, & Confrey, 2002). For example, one third grader's solution for finding the volume of a cylinder is displayed in Figure 12.5. The solution draws on the method described in Figure 12.4 of finding parts to compose whole units but refines the method to describe part-whole relations as fractional pieces, such as one fourth. Fractional pieces were then composed to estimate area units, for example, 1/4 + 1/2 + 1/4 = 1. After estimating the area of the circle in this manner, this student proposed finding volume by multiplying the estimated area by the height of the cylinder "to draw it [the area of the base] through how tall it is."

Battista (1999) followed the activity of three pairs of fifth-grade students as they predicted the number of cubes that fit in graphically depicted boxes. He found that student learning was affected both by individual activity and by socially constituted practices like collective reflection. Thus, traditional notions about trajectories of

development may need to be revised in light of more careful attention to classroom talk and related means of representing volume. Although the solution of the third student depicted in Figure 12.5 is unusual in the literature it was commonplace in a third grade w ere students had prolonged opportunities to explore mathematics of space and measure.

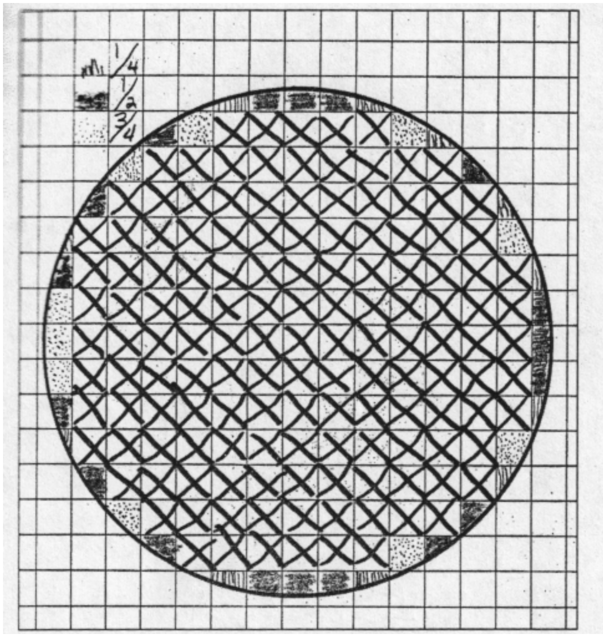

Figure 12.5. A third grader **first uses colors** to explicitly mark fractional pieces of units to approximate the area of the base of a cylinder and then finds volume as the product of area and height.

Angle Measure

Freudenthal (1973) suggested that multiple mathematical conceptions of angle should be entertained during the course of schooling. Henderson (1996) suggests three conceptions: (a) angle as movement, as in rotation or sweep; (b) angle as a geometric shape, a delineation of space by two intersecting lines; and (c) angle as a measure, a perspective that coordinates the other two. Mitchelmore (1997, 1998) and Lehrer et al. (1998b) found that students in the elementary grades develop separate mental models of angle as movement and angle as shape. In Mitchelmore's (1998) study, students in Grades 2, 4, and 6 increasingly perceived how different types of turning situations might be alike (e.g., those involving unlimited turning, like a fan, and those involving limited turning, like a hinge), but they rarely related these to situations involving "bends" or other aspects of intersecting lines. Lehrer et. al (1998b) asked children to find ways of measuring the bending in a hinge (with a sweep demonstrated from one position to another) and the bending in a bent pipe cleaner. Like the students in Mitchelmore's studies, students in Grades 1 through 5 rarely saw a relationship between these situations, but their measurement actions were very similar. Children most often chose to measure the distance between the jaws of the hinge and the ends of the pipe cleaners. In these static contexts ("bends"), students typically thought that angle measures were influenced by the lengths of the intersecting lines or by their orientation in space. The latter conception decreased with age, but the former was robust at every age (Lehrer et al., 1998b).

As noted for length, area, and volume measure, the tools employed or invented by students significantly affected their developing conceptions of angle measure. Studies of student learning with the Logo computer program generally confirm the existence of distinct models of angle as static or dynamic, respectively. Logo's Turtle geometry affords the notion of angle as a rotation, although students often confuse the interior and exterior (turtle) angles of figures traced by the Turtle. Nevertheless, with well-crafted instruction, tools like Logo mediate the development of angle measures as rotations (Lehrer, Randle, & Sancilio, 1989). However, students rarely bridge these rotations to models of the space in the interior of figures traced by the Turtle (e.g., Clements, Battista, Sarama, & Swaminathan, 1996). Simple modifications to Logo help students perceive the relationship between turns and traces (the path made by Logo's Turtle), and in these conditions students can use turns to measure static intersections of lines (Lehrer et al., 1989).

It remains a major challenge to design pedagogy to help students develop understanding of angle and its measure. Unlike the spatial structuring of linear dimensions (length, area, and volume), developing understanding of angle requires novel forms of representation that are perhaps not as prevalent in the culture (e.g., developing notions of turn, tracing a locus of a turning movement, relating turning movements to traces in environments like Logo). In addition, understanding angle involves the coordination of several potential models and integration of these models in a theory of their measure (Mitchelmore & White, 1998). Common admonitions to teach angles as turns usually fail because students rarely spontaneously relate situations involving rotations to those involving shape and form. As I have stated often in this tour of measure, the form of mediation (e.g., the tools, what students write, the models they explore) matters as much as the problems posed.

Expanding the Scope of Measure

Most research has been inspired by the Piagetian tradition of tight coupling between study of measure and study of space, and research findings indeed indicate that coming to understand measure is a productive means for learning about the structure of space. Nevertheless, the scope of measure often must extend beyond spatial extent, especially to support scientific reasoning. Although research here is sparser, it suggests that children's conceptions of measure can be extended to nonspatial realms.

Understanding the Natural World

Measurement is essential for developing an understanding of the natural world (Crosby, 1997). By quantifying and otherwise *mathematizing* nature (Kline, 1980), students can model the natural world, even at an early age. Although studies in science education often refer to the importance of measure, they generally do not "unpack" its conceptual foundations, so measurement is often viewed more as a matter of procedure than as a matter of conception. Yet a small number of studies suggest that developing concepts of measure can support better understanding of natural phenomena. In short, when children understand measure, its application to natural phenomena yields enhanced understanding in a manner reminiscent of the tight linkages between spatial extent and spatial structure noted previously. For example, third- and fifth-grade students who had the opportunity to develop understanding of spatial measure (e.g., length, area, volume) readily extended these understandings to the measure of mass. That is, with appropriate reminders from their teachers, they decided that units of mass should be identical, conventional, iterable, and so on. These understandings proved crucial for subsequent modeling material kind as a ratio of mass and volume—that is, density (Lehrer, Schauble, Strom, & Pligge, 2001). Similarly, third-grade children who had histories of learning about measure readily extended their ideas about unit to encompass rate, a ratio measure, to support reasoning about the growth (e.g., change in height per day) of plants (Lehrer, Schauble, Carpenter, & Penner, 2000). The challenges of ratio measures like these veer into more general considerations of children's understanding of rational number and multiplicative structure (e.g., Harel, Behr, Post, & Lesh, 1992). Unfortunately, little research addresses these potential relationships between measure and multiplicative structure.

Precision and Error

Much of the research about measurement explores precision and error of measure in relation to mental estimation (Hildreth, 1983; Joram, Subrahmanyam, & Gelman, 1998). To estimate a length, students at all ages typically employ the strategy of mentally iterating standard units (e.g., imagining lining up a ruler with an object). In their review of a number of instructional studies, Joram et al. (1998) suggest that students often develop brittle strategies closely tied to the original context of estimation. They suggest that instruction focus on children's development of reference points (e.g., landmarks) and on helping children establish reference points and units along a mental number line. Mental estimation would also likely be improved with more attention to the nature of unit, as suggested by many of the classroom studies reviewed previously.

Although mental estimation is one potential source of imprecision in measure, error is a fundamental quality of measure, a recognition that historically was quite troubling to scientists (Porter, 1986). Acts of measuring yield a range of estimates and, to the extent that errors are random, often a Gaussian ("normal") distribution. Hence, understanding of error is tied to conceptions of distribution. Conceptions of error are also central to scientific experimentation (Mayo, 1996). Schauble (1996), for example, found that fifth- and sixth-grade students who conducted experiments often confounded error and the variation due to a small, but reliable, effect of a variable. Perhaps they would not do so if they had opportunities to consider likely sources of error. Kerr and Lester (1976) suggest that instruction in measure should routinely encompass considerations of sources of error, especially (a) the assumptions (e.g., the model) about the object to be measured, (b) the choice of measuring instrument, and *(c)* the way the instrument is used (e.g., method variation). Variation among individuals is also commonly considered, especially in work in the social sciences. Recent work in classrooms explores children's ideas about some of these sources of error.

Varelas (1997) examined how third- and fourth-grade students made sense of the variability of repeated trials. Many children apparently did not conceptualize the differences among repeated observations as error, and children often suggested that fewer trials might be preferable to more. These conceptions seemed bound with relatively diffuse conceptions of representative values of a set of repeated trials. In a related study, Lehrer et al. (2000) found that with explicit attention to ways of ordering and structuring trial-to-trial variability, second-grade children made sense of trial-to-trial variation by suggesting representative (typical) values of sets of trials. Choices of typical values included "middle numbers" (i.e., medians) and modes, with a distinct preference for the latter.

In follow-up work with fourth-grade students, Petrosino, Lehrer, & Schauble (in press) further investigated children's ideas about sources and representations of measurement error. In one portion of the Petrosino et al. (in press) classroom teaching experiment, fourth-grade students measured the length of a pencil and the height of a flagpole. They represented each distribution of measurements, an accomplishment that involved ordering and putting like values in "bins." Students noticed differences in the comparative spread of each distribution (see Figure 12.6, in which the right panel depicts the pencil measures and the left panel, the flagpole measures) and readily attributed these differences to the relative precision of measure of the instruments available to conduct each measure (rulers vs. "height-o-meters"). The distribution obtained in each case was attributed primarily to individual differences in use of each instrument.

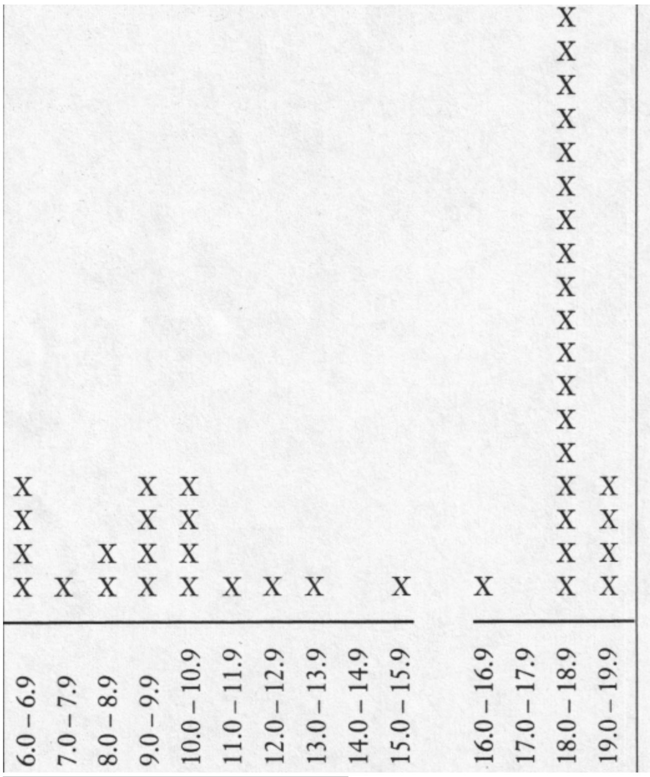

Figure 12.6. Distributions of fourth-grade students' measurements of the height of a flagpole (in m, left) and the length of a pencil (in cm, right).

When pressed to describe the differences in the spread that they perceived, students, with the assistance of their teacher, developed the notion of a "spread number" to quantify variation. One of the procedures invented by the class consisted of finding the median of the differences between observed measures and the typical measure (in this class, a median). Armed with these understandings, students went on to investigate a variety of sources of error and their consequences for a distribution of values. For example, Figure 12.7 displays the distributions of differences of observed values from the median for flagpoles measured with two different instruments (the left panel depicts errors with a handmade instrument, and the right panel depicts errors with a machined instrument). These distributions sparked examination of the comparative reliability of each instrument. This research suggests how relationships between variation and sources of error might be explored and elaborated in later grades. Such understandings are important foundations to the conduct of experiment and related forms of scientific explanation (Lehrer, Schauble, & Petrosino, 2001).

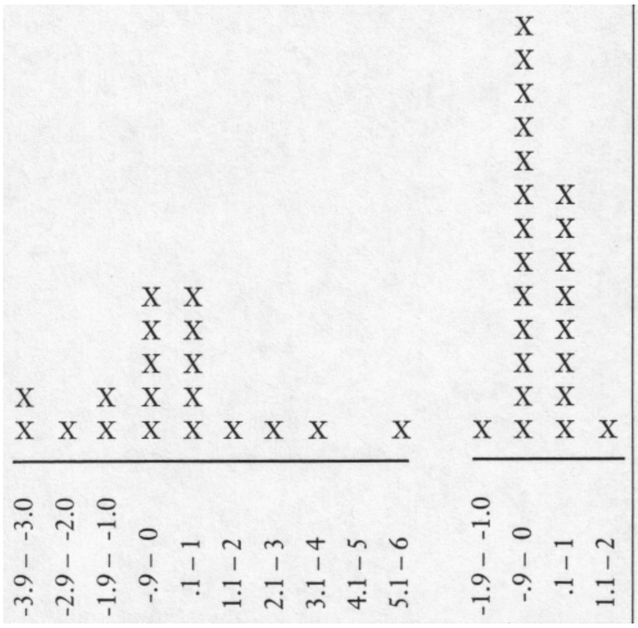

Figure 12.7. Distributions of differences of observed values from the median value of the height of a flagpole for measurements made with homemade (left) and machined (right) instruments.

Concluding Comments

Children are tacit measurers of nearly everything. Early and repeated experiences with cultural artifacts like rulers, and with the general epistemology of quantification that is characteristic of many contemporary societies, provides a fertile ground for developing mathematical understanding of measure. Developmental research suggests that children's conceptions of measure reflect a collection of emerging concepts whose coordination gradually unfolds as a network of relations, not a unitary concept of measure. Understanding some of these relationships spans the course of schooling. Classroom research points to the importance of helping children go beyond procedural competence to learn about the mathematical underpinnings of measure so that procedures and concepts are mutually bootstrapped. Measuring always involves doing, and such activity is always conducted in light of some model of the attribute being measured. Consequently, measuring a length, area, volume, or angle affords opportunities for developing ideas about the structure of space, such as its dimension, array, and curvature. Understanding structures like these, in turn, provides a platform for increased comprehension of number, as exemplified in spatial models like the number line or area models of rational number.

No clear-cut "best" sequence of instruction seems to exist in any domain of measure nor any reasonable list of prescriptions or proscriptions other than the need to avoid exclusive reliance on the development of procedural competence. As in other domains of mathematics, procedural competence (e.g., measuring with a ruler) can bootstrap conceptual development (e.g., inferring more general principles on the basis of the design of the ruler). Developing knowledge of effective procedures is a form of conceptual development, and developing concepts is aided and abetted by constructing and reflecting on ways to measure. Classroom studies emphasize the importance of helping children understand the rationale of familiar tools, such as rulers, and of finding productive ways for engaging children in the guided reinvention of such crucial concepts of measure as unit-attribute relations (e.g., length is not the measure of all things) and the very idea of unit. Teachers who understand the growth and development of student reasoning about measure are our collective "best bet" for generating productive learning about measurement.

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